MTH 605 - Midterm Solutions

1. A space X is said to be *contractible* if the identity map id_X on X is nulhomotopic.

   (a) Show that a space X is contractible if and only if it is homotopically equivalent to a one-point space.

   (b) Show that a retract of a contractible space is contractible.

   Solution. (a) Consider an $x_0 \in X$ such that $id_X \simeq c_{x_0}(\text{via}\, F)$. We claim that the constant map c_{x_0} is a homotopy equivalence between X and $\{x_0\}$. Clearly, $c_{x_0} \circ i_{x_0} = id_{\{x_0\}}$, where $i_{x_0} : \{x_0\} \to X$ is the inclusion. Furthermore, since $i_{x_0} \circ c_{x_0} \simeq c_{x_0}$, it follows from our hypothesis that $id_X \simeq i_{x_0} \circ c_{x_0}(\text{via}\, F)$.

   Conversely, suppose that X is homotopically equivalent to $\{x_0\}$. Then by assumption $c_{x_0} = i_{x_0} \circ c_{x_0} \simeq id_X$, from which it follows that X is contractible.

   (b) Let $x_0 \in A \subset A$, and let $r : X \to A$ be a retraction. Then we claim that $c_{x_0}|_A$ is a homotopy equivalence. As before, $c_{x_0}|_A \circ i_{x_0} = id_{\{x_0\}}$, where $i_{x_0} : \{x_0\} \to A$ is the inclusion. So it suffices to show that $id_A \simeq i_{x_0} \circ c_{x_0}|_A$. However, it is easily seen that $id_A \simeq i_{x_0} \circ c_{x_0}|_A(\text{via}\, r \circ F)$.

2. Let $f : S^1 \to S^1$ be a continuous map, and let $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$ be the homomorphism induced by f. Then the *degree* of f (denoted by $\deg(f)$) is an integer d such that $f_*([\alpha]) = d(\hat{\beta}_{f(1)}([\alpha]))$, where $[\alpha]$ is a generator of $\pi_1(S^1, 1)$ and β_x is the path in S^1 obtained by traversing along S^1 in the counterclockwise direction from a point $x \in S^1$ to 1.

   (a) Compute $\deg(f)$ for the following maps.

   (i) $f = c_1$.

   (ii) $f(z) = \bar{z}$.

   (iii) $f(z) = z^n$.

   (b) Show that $\deg(f \circ g) = \deg(f)\deg(g)$.

   (c) Let $f, g : S^1 \to S^1$ be continuous maps. Then show that $f \simeq g$ if and only if $\deg(f) = \deg(g)$.

   Solution. First, we note that $\deg(f)$ is independent of the choice of path (Verify this!). For simplicity, given $[\alpha] \in \pi_1(S^1, 1)$ and $f : S^1 \to S^1$, we define

$$[\alpha_f] := \hat{\beta}_{f(1)}([\alpha]) = [\bar{\beta}_{f(1)} * \alpha * \beta_{f(1)}].$$

(a) (i) When $f = c_1$, it follows that:

$$f_*([\alpha]) = [(c_1 \circ \alpha_f)] = [(c_1)_f],$$

which shows that $\deg(f) = 0$.

(ii) When $f(z) = \bar{z}$, we have:

$$f_*([\alpha]) = [(f \circ \alpha_f)] = [\bar{\alpha}_f] = -[\alpha_f],$$

where the bar over the α_f signifies complex conjugation. Thus, it follows that $\deg(f) = -1$.

(iii) When $f(z) = z^n$, we have:

$$f_*([\alpha]) = [(f \circ \alpha)] = [(\alpha_f)^n] = n[\alpha_f],$$

from which we can infer that $\deg(f) = n$.

(b) Let $\deg(f) = d_1$ and $\deg(g) = d_2$. Then given $[\alpha] \in \pi_1(S^1, 1)$, we have:

$$(f \circ g)_*([\alpha]) = f_*(g_*([\alpha])) = f_*(d_2[\alpha_g]) = d_2 f_*([\alpha_g] = d_2 d_1[\alpha_{f \circ g}],$$

from which the assertion follows.

(c) If $f \simeq g$, then $f_* = g_*$, from which it immediately follows that $\deg(f) = \deg(g)$. Conversely, suppose that $\deg(f) = \deg(g)$. We note that any continuous map $h : S^1 \to S^1$ can be viewed a loop in S^1 based at 1. (Too see this, verify the following fact: $\pi_1(S^1, 1)$ is in bijective correspondence with the homotopy classes of continuous maps $S^1 \to S^1$). Moreover, for any homotopy class $[\beta] \in \pi_1(S_1, 1)$, there exists a unique $d \in \mathbb{Z}$ such that $[\beta] = d[\alpha]$. Hence, as $\deg(f) = \deg(g)$, as loops they represent the same homotopy class $\pi_1(S_1, 1)$, from which it follows that $f \simeq g$.

3. Consider the torus T imbedded $\mathbb{R}^3$ whose points satisfy the quartic equation

$$\left(x^2 + y^2 + z^2 + r^2 - 1\right)^2 = 4r^2\left(x^2 + y^2\right),$$

where $r > 1$. Consider the antipodal identification on the torus T defined by $(x, y, z) \sim (-x, -y, -z)$ for each $(x, y, z) \in T$. Let $q : T \to K$ be induced quotient map and let K be the quotient space $T/\sim$.

(a) Show that $q : T \to K$ is a covering space.

(b) Show that there exists an index-two subgroup of the $\pi_1(K)$ that is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Solution. (a) First, we note that under the given embedding of T, the intersection of T with the $\{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ (xy-plane) in the disjoint union of the circles $S^1 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1, z = 0\}$ and $S^1_r = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = r^2, z = 0\}$. Moreover, T does not intersect the z-axis and any rotation of $\mathbb{R}^3$ about the z-axis leaves T invariant.

Under the antipodal identification $\sim$ on T each point point in $T \cap \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$ is identified with a unique antipodal point in $T \cap \{(x,y,z) \in \mathbb{R}^3 : z < 0\}$. Furthermore, each point on the circle S^1 (resp. S^1_r) is identified with a unique antipodal point on the S^1 (resp. S^1_r). Thus, the quotient space $K = T/\sim$ can also be described as being obtained by the antipodal identification on the two boundary components $S^1 \sqcup S^1_r$ of the hemitorus $T' = T \cap \{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$. (Show that $K \approx \mathbb{R}P^2 \# \mathbb{R}P^2$, the Klein bottle.)

Consider the quotient map $q : T \to K = T/\sim$ induced by the antipodal identification described above. Then any point $x \in K$ lifts to an antipodal pair $\{\tilde{x}, -\tilde{x}\}$ of points in T. Let $B(z,r)$ denote the open ball in $\mathbb{R}^3$ centered at z and radius r and let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the antipodal map. Choose $t < (r-1)/4$ and consider the open sets $U = B(\tilde{x}, t) \cap T$ and $A(U) = B(-\tilde{x}, t) \cap T$ in T. Then, clearly $U \cap A(U) = \emptyset$ and both U and $A(U)$ are mapped homeomorphically by q to an open set V in K containing x. Therefore, V is an evenly covered neighborhood of x, and the assertion follows.

(b) Following the notation in 3(a) above, let $H = q_*(\pi_1(T, \tilde{x}))$. Since $q : T \to K$ is a covering space, it follows from Lesson Plan 4.4 (iii)(b) that $\pi_1(K,x)/H \to \{\tilde{x}, -\tilde{x}\}$ is a bijection. Moreover, as q_* is injective, it follows that $H \cong \pi_1(T, (1,1)) \cong \mathbb{Z} \times \mathbb{Z}$. Thus, H is the required index-two subgroup of the $\pi_1(K,x)$ that is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

4. Let X be the quotient space obtained by identifying the circle $x^2+y^2 = 1, z = 0$ in the torus T (as in Problem 3 above) with the equator of the unit sphere S^2 centered at the origin $(0,0,0)$. Use the Seifert-Van Kampen theorem to compute the fundamental group of X.

   Solution. Let $A_1 = B(0, r-1/2) \cap X$ and $A_2 = T \cup (\{(x,y,z) \in \mathbb{R}^3 : z \in (-1/2, 1/2)\} \cap S^2)$. Following the notation in Problem 4, we have $X = A_1 \cup A_2$, where $A_1 \simeq S^2$, $A_2 \simeq T$, and $A_1 \cap A_2 \simeq S^1$. Let $x = (1,0,0)$ and $i_* : \pi_1(A_1 \cap A_2, x) \to \pi_1(X,x)$ and $j_* : \pi_1(A_1 \cap A_2, x) \to \pi_1(X,x)$ induced by the inclusions $i : (A_1 \cap A_2, x) \to (X,x)$ and $j : (A_1 \cap A_2, x) \to (X,x)$, respectively. Let β be the generator of $\pi_1(A_1, x) \cong \pi_1(S^1, x)$ represented by loop based at x in S^1 (the equator of S^2) that traverses once around S^1 in the counterclockwise direction, β' be the generator of $\pi_1(A_2, x) \cong \pi_1(T, x)$ represented by a loop based at x in T that traverses once around the logitudinal curve of T (along which T is identified with S^2 by construction), and let α' be the other generator of $\pi_1(T, x)$ represented by the loop based at x in T that traverses once around the meridional curve of T (perpendicular to the longitudinal curve at x) in the counterclockwise direction.

By construction, it is apparent that i_* is trivial, since S^2 is simply-connected, and $j_*(\beta) = \beta'$. By the Seifert-Van Kampen theorem, we get

$$\pi_1(X, x) = \pi_1(A_1, x) * \pi_1(A_2, x)/N,$$

where N is normally generated by $\{i_*(\alpha)j_*(\alpha)^{-1} : \alpha \in \pi_1(A_1 \cap A_2, x) \cong \pi_1(S^1, x)\}$. Hence, it follows that

$$\pi_1(X, x) \cong \langle \alpha', \beta' \rangle / \langle (\beta')^{-1} = (j_*(\beta))^{-1} \rangle \cong \langle \alpha' \rangle \cong \mathbb{Z}.$$

5. **(Bonus).** Show that for $n \geq 2$, S_n is not contractible.

   Solution. For $x_0 \in S^n$, suppose that $c_x \simeq id_{x_0}(\text{via}\, F)$. Then the map $r : D^{n+1} \to S^n$ defined by

$$r(x) = \begin{cases} x_0, & \text{if } \|x\| \leq 1/2, \text{ and} \\ F(\frac{x}{\|x\|}, 2\|x\| - 1), & \text{if } \|x\| \geq 1/2, \end{cases}$$

   defines a retraction of $D^{n+1} \to S^n$ (Verify this!), which is a contradiction. (Note that this proof assumes the following nontrivial fact: There exists no retraction from $D^{n+1} \to S^n$. The proof for $n = 1$ was discussed in class as part of Lesson Plan 2.5 (iv).)